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On the Action of a Fuzzy Group on a Fuzzy Set

***Akinola, L. S.**

Department of Computer Science, Federal University Oye-Ekiti, Ekiti State, Nigeria

Abstract

In this paper, we develop fundamental concepts required to extend the concept of group action on a set to fuzzy domain. We define product of a fuzzy set and a fuzzy group by using the idea of cartesian products of sets. We construct examples to demonstrate the defined concepts. We also discuss properties of the defined product of a fuzzy set and a fuzzy group as requisite to study of fuzzy group actions on fuzzy sets. Mathematics Subject Classification (2020). 08A99, 08C05, 22F05

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Introduction

In the past few decades, efforts have been on course to use fuzzy approach to model uncertainties that arise from real life phenomenon in the fields of engineering, science, social science, medical science, etc. Zadeh[12] in 1965, stem from his observation that many of the objects that require modelling in the real life situations do not have definite membership, introduced the theory of fuzzy set that allows every object to have certain degree of membership in real life set situations. He further developed various relevant operations on this fuzzy set and justified convincing relationships that exist between the introduced fuzzy set operations and conventional set operations. Rosenfeld[10] in 1971 introduced fuzzy group. He defined fuzzy group and developed many of its interesting properties. Ever since then, many other extensions of fuzzy sets have featured in literature that proposed further improvement in various directions. Notably among them are Ajmal and Thomas[1] that studied fuzzy subgroup and its quasinormality, Bhakat and Das[2] that developed the concept of interval valued fuzzy subgroup, and Liu[8] that worked on fuzzy invariant subgroup.

Generally, a group action on any set or algebraic structure is a group homomorphism of a group into the automorphism group of the structure. It depicts that the group *acts* on the set or algebraic structure. Action of a group on a set or structure is usually extended on structures built on the set. Group action generalizes group multiplication. If G is a group and X is an arbitrary set, a group action of an element $g \in G$ and $x \in X$ is a product, gx, living in X. Many problems in algebra may best be solved through group actions. Many works exist in literature building on algebraic group actions. Datuashvili[4-6] studied Witt's theorem for groups with action and free Leibniz algebras which was further extended to homological and homotopical properties of algebraic objects. Datuashvili and Sahan[7]defined and studied derived actions in the category of groups with

*Corresponding author: +2348073024003
Email address:

action on reduced group. This category, are identified to play a crucial role in the solution of two problems of semi-direct products in categories of groups with action category among others.

In this work, we define product of a fuzzy set and a fuzzy group and discuss its fundamental properties. We extend the concept of algebraic product of two fuzzy subsets of the same set to cover algebraic product of two fuzzy subsets of sets (not necessarily the same) and algebraic product of a fuzzy set and a fuzzy group in which one or more elements of the group is distinct from those of the set. We investigate the properties of the introduced concept as foundation towards defining the concept of fuzzy group acting on a fuzzy set.

Preliminaries

Algebraic Product: Let λ and μ be fuzzy sets of the set X, the algebraic product of λ and μ is defined as

$\lambda\mu(x) = \lambda(x)\mu(x)\ \forall\ x \in X.$

Clearly,

$(\lambda\mu)(x) \subset \lambda(x)\cap\mu(x)\ \forall\ x \in X.$

The dual of the algebraic product is the sum

$\lambda\oplus\mu = (\lambda^c\mu^c)^c = \lambda+\mu-\lambda\mu.$

Convex Combination: By the convex combination of two vectors f and g is usually meant a linear combination of f and g of the form

$\alpha f+(1-\alpha)g$, for which $0 \leq \alpha \leq 1$

The mode of combining f and g can be generalized to fuzzy sets in the following manner. Let α, β, and γ be arbitrary fuzzy sets. The convex combination of α, β, and γ denoted by $(\alpha,\beta;\gamma)$ and is defined as

$(\alpha,\beta;\gamma)=\gamma\alpha+\gamma^c\beta$

where γ^c is the complement of γ.

This can be written in terms of membership functions as:

$(\alpha,\beta,\gamma)(x)=\gamma(x)\alpha(x)+[1-\gamma(x)]\beta(x)$

One basic property of convex combination of α, β,and γ is expressed as

$\alpha\cap\beta\subset(\alpha,\beta;\gamma)\subset\alpha\cup\beta$

which is an immediate consequence of the inequalities

$\min[\alpha(x),\beta(x)]\leq\gamma(x)\alpha(x)+[1-\gamma(x)]\beta(x)\leq\max[\alpha(x),\beta(x)].$

Definition (2.1): Let X be a nonempty set. A fuzzy set defined on X is a function μ: $X\rightarrow[0,1]$.

For any $x\in X$, $\mu(x)$ represents the degree of membership of x in the fuzzy set of X. Thus, the nearer the value $\mu(x)$ to unity, the higher the grade of membership of x in X. A fuzzy set of a set X is empty if and only if its membership function is identically zero on X. Two fuzzy sets $\mu$and λ of a non- empty set X are said to be equal if and only if $\mu(x)=\lambda(x)\forall x \in X$.

Definition(2.2): Let X be a set and G be a group. A(left) action of G on X is a map $G\times X\rightarrow X$ given by $(g,x)\rightarrow gx$, where

1. $ex = x$ for all $x \in X$;
2. $(g_1g_2)x = g_1(g_2x)$for all $x \in X$ and all $g_1,g_2\in G$.

Definition (2.3): Let λ and μ be fuzzy subsets of the sets X and Y respectively. The Cartesian product of λ and μ defined as $\lambda\times\mu : X \times Y\rightarrow[0,1]$ such that

$(\lambda\times\mu)(x,y)=\min\{\lambda(x),\mu(y)\}\forall(x,y)\in X\times Y.$

Definition (2.4): Let μ be a fuzzy set in a set G. Then, the level subset μ_t is defined as:

$\mu_t=\{x \in G: \mu(x)\geq t\}$ for $t\in[0,1]$.

Definition (2.5): Let μ be a fuzzy set in a group G. Then, μ is said to be a fuzzy subgroup of G, if the following hold:

(i) $\mu(xy)\geq\min\{\mu(x),\mu(y)\}\ \forall x,y\in G$;
(ii) $\mu(x^{-1})=\mu(x)\ \forall x\in G$.

Definition (2.6) Let μ be a fuzzy sub group of G. Then, the level subset μ_t, for $t\in \mathrm{I\ m}\ \mu$ is a subgroup of G and is called the level subgroup of G.

The Results

We adopt the concept of Cartesian product of two fuzzy sets as it exists in literature to define Cartesian product of a fuzzy group and a fuzzy set.

Definition (3.1): Let G be a group and X a nonempty set. Let λ be a fuzzy subgroup of G and μ be fuzzy sub set of the set X. The Cartesian product of λ and μ defined as $\lambda\times\mu : G\times X\rightarrow[0,1]$ such that $(\lambda\times\mu)(g,x)=\min\{\lambda(g),\mu(x)\}\ \forall(g,y) \in G\times X$.

Example (3.2): Let G= {a,b,c,d,e,f} represents the symmetries of an equilateral triangle. It is easy to see from the following Cayley table that G is a group with respect to the operation of composition of mapping.

o	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	a	d	c	f	e
c	c	f	a	e	d	b
d	d	e	b	f	c	a
e	e	d	f	b	a	c
f	f	c	e	a	b	d

Define a fuzzy subset $\gamma_G : G\rightarrow[0,1]$ as:
$\gamma(a) = 0.8, \gamma(e) = \gamma(f) = 0.7, \gamma(b) = \gamma(c) = \gamma(d) = 0.5$.

It is easy to see that $\gamma_G : G\rightarrow[0,1]$ is a fuzzy group.
Let X= {x,y,z,w} be a set. Define $\mu X : X\rightarrow[0,1]$ be defined as:

$\mu(x)=0.8, \mu(y)=0.5, \mu(z)=0.7, \mu(w)=0.9$.

$\gamma_G\times\mu_X$ is obtained in the following table

$\lambda_G\times\mu_X$	μ(x)=0.8	μ(y)=0.5	μ(z)=0.7	μ(w)=0.9
λ(a)=0.8	0.8	0.5	0.7	0.8
λ(b)=0.5	0.5	0.5	0.5	0.5
λ(c)=0.5	0.5	0.5	0.5	0.5
λ(d)=0.5	0.5	0.5	0.5	0.5
λ(e)=0.7	0.7	0.5	0.7	0.7
λ(f)=0.7	0.7	0.5	0.7	0.7

Proposition (3.3): Let λ be a fuzzy group of G and μ a fuzzy set of a set X. For any $x \in X$, $\lambda\times\mu(e,x) = \mu(x)$ if and only if $\lambda(e) \geq \mu(x)$ where e be the identity element of a group G.

Proof: Straight forward

For any two fuzzy sets α and β of the same set, the algebraic product of α and β has been defined as $\alpha\beta(x) = \alpha(x)\beta(x)\ \forall\ x \in X$.

If X is a set and G a group such that every $g \in G$, $g \in X$, then it will be a straightforward matter to define the product of fuzzy group λ of G and a fuzzy set φ of X.

It is interesting to ask if it will be possible to define an algebraic product of two fuzzy set of different sets or as an extension, if it will be possible to define an algebraic product of a fuzzy set and a fuzzy group G where G contain some other elements apart from the elements in which the fuzzy set is defined on.

The answer here is yes. This is based on the philosophy of the fuzzy set itself that allows any element to be a member of any set with varying degree of membership.

We hereby extend the concept of algebraic product of two fuzzy subsets of the same set to cover algebraic product of two fuzzy subsets of sets (not necessarily the same) and algebraic product of a fuzzy set and a fuzzy group in which one or more elements of the group is distinct from those of the set.

Definition (3.4): Let λ be a fuzzy subgroup of a group G and φ a fuzzy subset of a set X. The algebraic product of λ and φ denoted as $\lambda\cdot\varphi$ is defined as

$$(\lambda\cdot\varphi)(x)=\begin{cases} 0 & \text{if } x\in G \text{ or } x\in X,\\ \gamma(x)\cdot\varphi(x) & \text{if } x\in G \text{ and } x\in X\end{cases}$$

The definition follows accordingly if G is just a set and not a group

Proposition (3.5): Let λ_1 and λ_2 be fuzzy subgroups of the group G,and μ a fuzzy set of a set X, then for any $(g,x)\in G\times X$,

$(\lambda_1\cdot\lambda_2)\times\mu(x,y)=\lambda_1\cdot(\lambda_2\times\mu)(x,y)$

Proof:

$$\begin{aligned}(\lambda_1\cdot\lambda_2)\times\mu)(x,y) &= (\lambda_1\times\mu)(x,y)\cdot(\lambda_2\times\mu(x,y)\\ &= \min[\lambda_1(x),\mu(y)]\cdot\min[\lambda_2(x),\mu(y)]\\ &\leq\min[\lambda_1(x)\cdot\lambda_2(x),\mu(y)]\\ &= \min[\lambda_1(x)\lambda_2(x),\mu(y)]\\ &= \min[\lambda_1\cdot\lambda_2(x),\mu(y)]\\ &= \lambda_1\cdot\lambda_2\times\mu(x,y)\end{aligned}$$

Corollary (3.6): Let λ_1 and λ_2 be fuzzy subgroups of the group G,and μ a fuzzy set of a set X, then for any $(g,x)\in G\times X$,

$(\lambda_1\cdot\lambda_2)\times\mu(g,x)\leq(\lambda_1\cdot\lambda_2)\times\mu(e,x)$

Proof: Follows directly from proposition (3.5) and the fact that $\lambda(x)\leq\lambda(e)$, for all g, e $\in$ G where e is the identity element of G.

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