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Bicyclic work done on a star-like finite chain semigroup

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ABSTRACT

This paper compares the work of bicyclic products and idempotent product chains on the star-like operator $|y-\alpha x| \leq |\alpha y - x|$ to demonstrate potential connections between semigroup and group theory. This research advances our understanding of star-like transformation semigroups and their algebraic features, particularly in terms of bicyclic and idempotent products. The findings add to previous understanding by obtaining the arithmetic and polynomial sequential patterns. We show that the composition of any star-like transformations produces the bicyclic products, such that $|T\omega_n^*| = \binom{n+2(\alpha_i^*-\beta_j^*)}{3(\alpha_i^*+\beta_j^*)}$, for all $\alpha_i^*, \beta_j^* \in T\omega_n^*$. The research contributes to the larger discipline of abstract algebra by introducing new results and approaches for examining algebraic structures.

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Introduction

The study of various finite semigroups of transformations makes a substantial contribution to semigroup theory, just as research on finite symmetric and alternating groups. Algebraic and combinatorial properties of transformation semigroups have been studied over time, with numerous notable findings resulting from the work of certain well-known mathematicians [1, 2, 4, 5, 6, 7, 8]. Their study has yielded valuable tools that may be applied to a wide range of combinatorial mathematics problems. The semigroup of star-like transformations is closed under composition, and the star-like disk points that belong to the semigroup of star-like transformations are said to be convex [9]. By establishing the star-like bicyclic products as a basic model, this research work aims to extend the star-like operator, which was recently created by [10]. The bicyclic transformation semigroup has received a lot of interest in recent years for its applications in cryptography, coding theory and automata theory. Research on finite full transformation semigroups provides significant benefits to semigroup theory. Numerous research studies have been conducted to investigate major properties of several of these semigroups. [3] proved that $|E(O_n)| = F_{2n}$, where F_n represents the nth Fibonacci number. It is worth noting that this research was inspired by the findings of [3] and [9].

Basic preliminary

Definition 1: The process in which a transparent A4 sheet is folded in a star-like form and then some star-like sequences and prisms are represented is defined as the star-like folding principle.

Definition 2 [8]: Star-like face (H^*) is the star-like flat or curving surfaces of a star-like transformation semigroup.

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Definition 3 [8]: Star-like vertex is the star-like transformation corner where two or more star-like edges meet.

Definition 4 [8]: The **s**tar-like edge (G^*) is a star-like disk point where two or more star-like faces meet.

Definition 5 [9]: The star-like disk point represents the order of the unique element that maps to itself, denoted by $D_k^*(\beta_n^*) = |\mu(\beta^*)| = |\{u\epsilon \mathrm{Z}_n^+ : u\mu(\beta^*) = u\}|$.

Definition 6: Given that $B\omega_n^*$ is a star-like full transformation semigroup, any star-like transformation $\beta\omega_n^* \epsilon B\omega_n^*$ is said to be a star-like bicyclic product if there exists a star-like disk point such that $x_i\beta\omega_{n+1}(B\omega_n^*)\overset{\Delta}{\nabla}x_j\beta\omega_n^*(B\omega_n^*)$.

The bicyclic products work done on a star-like finite chain

We are most interested in the bicyclic products of finite chains and the star-like transformation semigroup. Figures 1 and 2 model the work done by cyclic and bicyclic products on star-like finite chains of transformation.

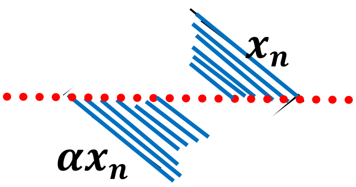

Fig. (1): Cyclic star − like workdone

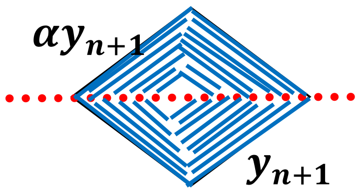

Fig. (2): Bicyclic star − like workdone

Assuming $\alpha \,\epsilon\, B\omega_n^*$ is bicyclic star-like work performed on a finite chain of star-like transformations, the above models show that $y_{n+1}\alpha x_n\overset{\Delta}{\nabla}\ x\alpha y_{n+1}$, if

$$|\Delta y\alpha x_n| \le |\alpha x_i - y_{n+1}|, \qquad (1)$$

$$|\nabla x\alpha y_{n+1}| \le |\alpha y_i - x_{n+1}|, \qquad (2)$$

for all $x, y \,\epsilon\, X_n$ and $\alpha x_n, \alpha y_{n+1} \,\epsilon\, B\omega_n^*$ .Then, adopting the star-like folding principle on a bicyclic star-like finite chain of transformation, we have

$$xy \le \alpha x\, \alpha y : \alpha y_{n+1} \le \alpha x_{n+1}. \qquad (3)$$

Main results

The research's main findings are as follows:

Theorem 1: The set of all star-like transformations of a finite star-like bicyclic is a semigroup

Proof:

We need to show that $\alpha T\omega_n^*$ is:

i. Star-like closed
ii. Star-like associative

For the star-like closed property: If $\alpha^*, \beta^* \,\epsilon\, \alpha T\omega_n^*$, then,

$$|x - y| \ge |x\alpha^* - y\alpha^*| \text{ and } |x - y| \ge |x\beta^* - y\beta^*|$$

We need to show that $|x - y| \ge |x(\alpha^*\beta^*) - y(\alpha^*\beta^*)|$.

Now,

$$|x(\alpha^*\beta^*) - y(\alpha^*\beta^*)| = |(x\alpha^*)\beta^* - (\alpha^* y)\beta^*|$$
$$\le |(x\alpha^*) - (y\beta^*)|$$
$$\le |x - y|$$

Therefore, $\alpha^*, \beta^* \,\epsilon\, \alpha T\omega_n^*$.

For the star-like associativity property, consider $|x - y| \le |x\alpha^*| - |x\beta^*|$, we get

$$|x - y| \ge |x\beta^*| - |y\beta^*|.$$

Then $|x| \le |x\alpha^* - x\beta^*| - |y| \le |y\alpha^* - y\beta^*|$
$\le |x\alpha^* - y\alpha^*| - |x| \ge |y\beta^* - x\beta^*|$,

Therefore,

$$|x - y|\overset{\Delta}{\nabla}|x(\alpha^*\beta^*) - y(\alpha^*\beta^*)|.$$

Theorem 2: The following statements are true for any bicyclic star-like transformation $\gamma^*\epsilon T\omega_n^*$

i. Any $\gamma^*\epsilon T\omega_n^*$ has a star-like supremun $(\gamma^*)S\{up\}$
ii. $|T\omega_n^*| = \cup_{i=2}^n \binom{n + 2(\alpha^*)}{3(\alpha^*)}$
iii. $f(n, c(\alpha^*), u) = (2^{(n-c(\alpha^*)+u)} - e^{u_0})$

Proof:

$$\boldsymbol{i \to ii}$$

Assume $f(n, c(\alpha^*), u)$ are star-like transformations with a distinct non-negative integer $\mathrm{Z}_n^+ = \{1,2,3, \dots\}$ such that $\cup(\gamma^*)S\{up\}$ contains a maximum image set of $\gamma^*\epsilon T\omega_n^*$. Then $i, j\epsilon \mathrm{Z}_n^*$, $I(\alpha_i\beta_j)$ can be chosen in $\binom{(\gamma^*)S\{up\}}{n_{ij}}$ ways.

Since $\cap_{i=j}^n \alpha_i^*\beta_j^*$ is a star-like bijective mapping, $T\omega_n^*$ is bijective.

If $I(\alpha_i^*\beta_j^*) = 0$

$$n|c(\alpha^*\beta^*)| = 1. But$$

If $n = c(\alpha_i^*\beta_j^*) = u\epsilon(\gamma^*)S\{up\}$

Such that: $|c(\alpha_i^*\beta_j^*)| - (\gamma^*)S\{up\} = 0|T\omega_n^*|$

For each value of $\alpha_i^*\beta_j^*\epsilon T\omega_n^*$ where $|T\omega_n^*| = \binom{n + 2(\alpha_i^* - \beta_j^*)}{3(\alpha_i^* + \beta_j^*)}$.

$$\boldsymbol{ii \to iii}$$

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If $\alpha^*\beta^*\epsilon T\omega_n^*$ is a star-like bicyclic, there exist finitely many star-like elements such that:

$D(\alpha_i^*\beta_j^*) = I(\alpha_i^*\beta_j^*)$, where $\alpha_i^*, \beta_j^* / \gamma_n^* \cap S\{up\}_n$ form a total of:

$$\mathrm{U}_{i=j}^{n}\begin{pmatrix} 2^{(n-c(\alpha_i^*\beta_j^*)+1} \\ 2^{(n-u)+1} - e^{u_0} \end{pmatrix}\begin{pmatrix} n + 2c(\alpha_i^*\beta_j^*) \\ 3(\alpha_i^*\beta_j^*) \end{pmatrix}.$$

$$\boldsymbol{iii \to i}$$

Since $(\gamma^*)S\{up\}$ is the star-like supremun element in $I(\alpha_i\beta_j)$, then with the composition of a star-like transformation in

$\gamma_n^* S_n\{up\} = \begin{pmatrix} e^{u_1} & e^{u_2} & e^{u_n} \\ c(\alpha_i\beta_j)^1 & c(\alpha_i\beta_j)^2 & c(\alpha_i\beta_j)^n \end{pmatrix}$, where there exists another star-like element

$$\gamma_n^* S_n\{inf_n\} \epsilon \ker \mathrm{c}(\alpha_i\beta_j)$$

with

$$\gamma_n^* S_n\{inf_n\} < e^{u_n} < I(\alpha_i\beta_j)$$

Obviously, both $\gamma_n^* S_n\{inf_n\}$ in ker $(c(\alpha_i\beta_j)$ are star-like bijective, hence,

$$\gamma_n^* S_n\{up\} \cap \gamma_n^* S_n\{inf_n\} < e^{u_0}.$$

Theorem 3: Let $X_n = \{x_1, x_2 \dots, n\}$ *be a unique non-negative integer*, then $x_i^2 = x_i$ if $B\omega_n^* \subseteq T\omega_n^*$ is dual Simple such that $x_i = p_i^m q_j^n$ whenever $m - 1 = n + 1$.

Proof:

Suppose that I is a two-sided ideal of $B\omega_n^*$ and $I \neq B\omega_n^*$.

Consider any element $x_i \epsilon I$.

Since I is a two-sided ideal for any $y_j \epsilon B\omega_n^*$, we have $y_j x_i \epsilon I$ and $x_i y_j \epsilon I$ for $i, j = 1,2,3, \dots, n$.

Particularly, for $y_j = q_j^n$, we get

$$q_j^n x_i = q_j^n p_i^m \epsilon I.$$

Choose n large enough so that $m = n$.

Then $q_j^n p_i^n = 1$and $1 \epsilon I$.

But if $1 \epsilon I$, then for any $Z \epsilon B\omega_n^*$, $Z = Z{:}\, 1 \epsilon I$.

So $I = B\omega_n^*$, which contradicts the assumptions that $I \neq B\omega_n^*$.

Hence, $B\omega_n^*$ has no proper two-sided ideal. Now, given a star-like disk-point, say $x_i\ \epsilon\ e\gamma_n^*$ to be a bicyclic product if $x_i^2 = x_i$. Let $x_i = p_i^m q_j^n$, then,

$$x_i^2 = (p_i^m q_j^n)(p_i^m q_j^n) = p_i^m(q_j^n p_i^m) q_j^n,$$

if

$$p_i^m q_j^n = p_i^m (p_i^m q_j^n) q_j^n.$$

But we discovered that $q_j^n p_i^m = 1$ only when $m = n$, which implies that $p_i^m q_j^n = 1$.

Hence $x_i = 1$.

Consider $y_j = p$.

Then $x_i p = p_i^m q_j^n p = p_i^{(m-1)} q_j^n$,

and $p x_i = p p_i^m q_j^n = p_i^{(m+1)} q_j^n$.

For these to be true or equal, we set $m - 1 = m + 1$, which is a contradiction that unless $m = 0$. Similarly, considering $y_j = q$, we conclude that $n = 0$, thus $x_i = 1$, and the centre $Z(B\omega_n^*)$ is a star-like disk point.

Remark 1: Given $e\omega_2^* = 2$ the star-like idempotent product element, the star-like bicyclic product element $B\omega_2^* = 2$, such that $|T\omega_2^*| = 3$ and $n - 1 = 2 - 1 = 1$.

Therefore, $B\omega_2^* + e\omega_2^* = |T\omega_2^*| + n - 1 = 4$.

Remark 2:.Suppose β_n^*, α_n^*, $\gamma_n^*\ \epsilon\ e\omega_n^*{:}\ \alpha_n^* = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \beta_n^* = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ *and* $\gamma_n^* = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$. Using the star-like bicylic product operator in the Equation, we observed that β_n^* It is not a star-like product. Hence, the work done by the star-like bicyclic product is an acyclic which is less than but not equal to its generators.

Conclusion

This work looked at the bicyclic and cyclic products of star-like transformations and discovered fresh insights into the algebraic structures of these products, where the transformation of two star-like may or may not yield a star-like finite chain. The results reveal that bicyclic products have unique qualities, such as the ability to build sequential chains with polynomial patterns, indicating that the sequence's ratios and differences are not constant. Table 1 shows various recursive relations derived from the bicyclic and cyclic products

Table 1: Work done by bicyclic products on star-like finite chains summary

$B_n^*\omega_n{}^*$ Functions	Results summary
$(R_k^*) = \lvert Im(\alpha^*)\rvert$	$a_n = n^2 - 3n + 4$
$C^+(\beta_n^*)$	$a_n = 288n^2 - 840n + 556$
$R_s^*(\beta_n^*)$	$a_n = 288n^2 - 840n + 556$
$K^+(\beta_n^*)$	$a_n = n^2 + 5$
$D_k^*(\beta_n^*)$	$a_n = -3n^2 + 22n - 15$
$J_t^*(\beta_n^*)$	$a_n = An^3 + \left(\frac{1995}{10} - 6A\right)n^2 + \left(11A - \frac{5855}{10}\right)n + (390 - 6A)$
$K^-(\beta_n^*)$	$a_n = 116n^2 - 329n + 217$
$P_t^*(\beta_n^*)$	$a_n = 130n^2 - 366n + 240$
$D_f^*(\beta_n^*)$	$a_n = -43n^2 + 185n - 140$

Conflict of interest

All authors have declared that there are no financial or other conflicts of interest in this manuscript.

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